THE CHINESE UNIVERSITY OF HONG KONG
MATH3270B
HOMEWORK4 SOLUTION

Question 1:

(1) Since we have

$$\lambda I - A = \begin{pmatrix} \lambda-1 & -1 & -1 \\ -2 & \lambda-1 & 1 \\ 3 & -2 & \lambda-4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & \lambda-1 \\ -1 & \lambda-1 & -2 \\ -\lambda+4 & -2 & 3 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & -1 & \lambda-1 \\ 0 & \lambda-2 & \lambda-3 \\ 0 & -\lambda+2 & 3+(\lambda-1)(\lambda-4) \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\lambda-2)^3 \end{pmatrix} \tag{1}$$

Hence, the Jordan form of A is

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \tag{2}$$

(2) Since the Jordan form of A is $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. We have that $\lambda = 2$ is the triple root of A, hence, we substitute $\lambda = 2$ into the matrix $\lambda I - A$ and deduce that the corresponding eigenvector is $\xi = (0, 1, -1)^T$. Therefore, first solution of that system is $x_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}$. Next, we assume that the second solution having the form $x_2 = \xi t e^{2t} + \eta e^{2t}$, where ξ and η satisfy:

$$\begin{cases} (A-2I)\xi = 0 \\ (A-2I)\eta = \xi \end{cases} \tag{3}$$

We solve (3) and deduce that $\xi = (0, 1, -1)^T$, $\eta = (1, 1, 0)^T$. Hence, the second solution is $x_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$. Similarly, we assume the third sol having the form $x_3 = \xi \frac{t^2}{2} e^{2t} + \eta t e^{2t} + \zeta e^{2t}$, where ξ, η, ζ satisfy:

$$\begin{cases} (A-2I)\xi = 0 \\ (A-2I)\eta = \xi \\ (A-2I)\zeta = \eta \end{cases} \tag{4}$$

We solve the (4) and deduce that $\xi = (0, 1, -1)^T$, $\eta = (1, 1, 0)^T$, $\zeta = (2, 0, 3)$. Therefore, the general solution of the system is

$$x = \varphi(t)x_0 = e^{2t}\begin{pmatrix} 0 & 1 & t+2 \\ 1 & t+1 & \frac{t^2}{2}+t \\ -1 & -t & -\frac{t^2}{2}+3 \end{pmatrix}x_0 \tag{5}$$

Question 2:

(1) Similarly, we have:

$$\begin{aligned} \lambda I - A = \begin{pmatrix} \lambda-5 & 3 & 2 \\ -8 & \lambda+5 & 4 \\ 4 & -3 & \lambda-3 \end{pmatrix} \to \begin{pmatrix} 2 & 3 & \lambda-5 \\ 4 & \lambda+5 & -8 \\ \lambda-3 & -3 & 4 \end{pmatrix} \\ \to \begin{pmatrix} 2 & 3 & \lambda-5 \\ 0 & \lambda-1 & -2\lambda+2 \\ 0 & -3\lambda+3 & \lambda^2+8\lambda-7 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & (\lambda-1)^2 \end{pmatrix} \end{aligned} \tag{6}$$

Hence, the Jordan form of A is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

(2) Similarly, we substitute $\lambda = 1$ into A and deduce that the eigenvectors are $\xi_1 = (1, 0, 2)^T$, $\xi_2 = (0, 2, -3)^T$. Hence, we get two linearly independent solution $x_1 = \xi_1 e^t$, $x_2 = \xi_2 e^t$. And we assume that the third solution having the form $x_3 = \xi t e^t + \eta e^t$ and ξ, η satisfy

$$\begin{cases} (A-2I)\xi = 0 \\ (A-2I)\eta = \xi \end{cases} \tag{8}$$

We solve (8) and deduce that $\xi = (2, 4, -2)^T$, $\eta = (0, 0, 1)^T$. Hence the general solution of system is

$$x = \varphi(t)x_0 = e^t\begin{pmatrix} 1 & 0 & 2t \\ 0 & 2 & 4t \\ 2 & -3 & -2t+1 \end{pmatrix}x_0 \tag{9}$$

Question 3:

(1) It's easy to deduce that $A = \begin{pmatrix} 0 & 1 \\ 0 & \omega^2 \end{pmatrix}$. And we find that: $A^2 = \begin{pmatrix} 0 & -\omega^2 \\ 0 & \omega^4 \end{pmatrix}$, $A^3 =$

$\begin{pmatrix} 0 & \omega^4 \\ 0 & -\omega^6 \end{pmatrix}$. Hence, we deduce that:

$$A^n = \begin{pmatrix} 0 & (-\omega^2)^{n-1} \\ 0 & (-\omega^2)^n \end{pmatrix}$$

Next, since $\sum_{n=1}^{+\infty} \dfrac{(-\omega^2)^{n-1}t^n}{n!} = \dfrac{e^{-\omega^2 t} - 1}{\omega^2}$, and $\sum_{n=1}^{+\infty} \dfrac{(-\omega^2)^n t^n}{n!} = e^{-\omega^2 t} - 1$, we deduce that:

$$e^{At} = I + \sum_{n=1}^{+\infty} \frac{(At)^n}{n!} = I + \begin{pmatrix} 0 & \dfrac{e^{-\omega^2 t} - 1}{\omega^2} \\ 0 & e^{-\omega^2 t} - 1 \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{e^{-\omega^2 t} - 1}{\omega^2} \\ 0 & e^{-\omega^2 t} \end{pmatrix} \tag{10}$$

Now, the initial data is $x_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$ We deduce finally that the sol is $x = e^{At}x_0 = \begin{pmatrix} y_0 + y_1 \dfrac{e^{-\omega^2 t} - 1}{\omega^2} \\ y_1(e^{-\omega^2 t} - 1) \end{pmatrix}$.

(2) First we consider the corresponding homogeneous equation. Here $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$. It's easy to deduce that the eigenvalues are $\lambda_1 = 1, \lambda_2 = -1$, and the corresponding eigenvectors are $\xi_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $\xi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence, the general sol of homogeneous equation is $x = C_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$. Then we assume that the particular of the inhomogeneous equation is $x^* = \varphi(t)u(t)$ where $\varphi(t)$ is the fundamental matrices and $\varphi(t) = \begin{pmatrix} -3e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix}$. By the method of variation of parameters, $u(t)$ satisfies the equation: $\varphi(t)u^{'}(t) = g(t)$,.i.e.

$$\begin{pmatrix} -3e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} \begin{pmatrix} u_1^{'} \\ u_2^{'} \end{pmatrix} = \begin{pmatrix} e^t \\ t \end{pmatrix} \tag{11}$$

We write (11) in scalar form,

$$\begin{cases} -3e^t u_1^{'} + e^{-t} u_2^{'} = e^t \\ e^t u_1^{'} - e^{-t} u_2^{'} = t \end{cases} \tag{12}$$

that is,

$$\begin{cases} u_1^{'} = -\dfrac{1}{2} - \dfrac{t}{2e^t} \\ u_2^{'} = \dfrac{1}{2}(-3te^t - e^{2t}) \end{cases} \tag{13}$$

We solve the (13) and deduce that:

$$\begin{cases} u_1 = -\frac{1}{2}t + \frac{1}{2}(t+1)e^{-t} + c_1 \\ u_2 = -\frac{1}{4}e^{2t} - \frac{3}{2}(te^t - e^t) + c_2 \end{cases} \tag{14}$$

Hence that general sol is

$$\begin{aligned} x = \varphi(t)u(t) &= \begin{pmatrix} -3e^t & e^{-t} \\ e^t & -e^{-t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}t + \frac{1}{2}(t+1)e^{-t} + c_1 \\ -\frac{1}{4}e^{2t} - \frac{3}{2}(te^t - e^t) + c_2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} e^t + \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} te^t + \begin{pmatrix} -3 \\ 2 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{aligned} \tag{15}$$

Question 4:

I think the problem is obvious. Since for equation (1), we have Abel's formula:$W(y_1, y_2) = Ce^{-\int p(t)dt}$. For the system (2), we also have the Abel's formula $\hat{W}(\phi_1, \phi_2) = \hat{(}C)e^{\int tr(A)dt} = \hat{C}e^{-\int p(t)dt}$. Hence, the conclusion follows.(tr(A) means the trace of A.)